Partial Fractions Problem Solving Process

1. Check if rational expression is in lowest terms, i.e. $\text{degree}(\text{numerator}) < \text{degree}(\text{denominator})$. If not, do long division to simplify the rational expression.
2. Check to see if the denominator in the rational expression is factorable and factor it! Use calculators, Wolfram Alpha, or long/synthetic division to factor
3. Set rational expression equal to the sum of different factors in the denominator. Fill in the numerators with variables accordingly:
   a. The numerator should be a polynomial of **one** degree less than the denominator (in factored form. (Convention is to start with A, B, C,… as unknown variables.)
   b. If a factor in the denominator has a power greater than one, i.e. $(x+5)^2$, repeat (a) continuously, each time decreasing the power by one until the factor has a power of one.
4. With the rational expression now equal to a sum of other rational expressions with unknowns, find a common denominator for all, multiply through by it.
5. Solve for the unknowns by coefficient comparison – set the coefficients of x equal, the coefficients of x^2 equal, etc and solve.
6. With the unknowns solved for, integrate these smaller rational expressions using known integration techniques to finish the process.

Example:
  Integrate by Partial Fractions.

$$\int \frac{x^4 + 4x^3 - x^2 - 2x + 9}{x^3 + x^2 - 5x + 3} dx$$

<u>Step 1:</u> This is not in lowest terms as degree of the numerator (4) is GREATER than the degree of the denominator (3) so we must do long division: (shown by WolframAlpha)

So the integral becomes

$$\int \frac{x^4 + 4x^3 - x^2 - 2x + 9}{x^3 + x^2 - 5x + 3} dx = \int \left[(x+3) + \frac{x^2 + 10x}{x^3 + x^2 - 5x + 3} \right] dx = \int (x+3) dx + \int \frac{x^2 + 10x}{x^3 + x^2 - 5x + 3} dx$$

We know how to integrate the first integral, so we will focus on the second part for now.

Step 2: We check to see if the denominator is factorable. We can simply plug this into our calculators or Wolfram Alpha, HOWEVER should we not have these tools available, remember try more elementary tactics: look at the factors of your degree zero term (the term without an x) and try synthetic division with these numbers. Here we have $\pm 1, \pm 3$ to try. The division is omitted, but the denominator factors into $(x-1)^2(x+3)$

Step 3: Continuing in the process we get:

$$\frac{x^2+10x}{(x-1)^2(x+3)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+3}$$

Note: As we had a factor of $(x-1)^2$, we had to employ part (b) here. Also notice that since all the factors were of degree one, all the numerators are of degree zero!

Step 4: Now for the Algebraic intensive step. Our common denominator will be the factored form on the left side of the above equation -- $(x-1)^2(x+3)$. **NOTE: Will this always be the case??** So we multiply through by this to ALL terms:

$$x^2+10x = A(x+3) + B(x-1)(x+3) + C(x-1)^2$$
$$x^2+10x = Ax+3A + Bx^2 + 2Bx - 3B + Cx^2 - 2Cx + C$$

Step 5: Now we do coefficient comparison: set all degree 0 coefficients equal, degree 1 coefficients equal and degree 2 coefficients equal, and then solve the resulting system of equations:

$$(x^2) \quad 1 = B+C \quad \Rightarrow B = 1-C$$
$$(x) \quad 10 = A+2B-2C$$
$$(x^0) \quad 0 = 3A-3B+C$$

Using the last equation,
$$0 = 3A - 3(1-C) + C = 3A - 3 + 4C$$
$$A = 1 - \frac{4}{3}C$$

Using the second equation,
$$10 = \left(1-\frac{4}{3}C\right) + 2(1-C) - 2C = 3 - \frac{16}{3}C$$
$$7 = -\frac{16}{3}C \Rightarrow C = -\frac{21}{16}, B = \frac{37}{16}, A = \frac{11}{4}$$

Step 6: Now we integrate our result:

$$= \int (x+3)dx + \int \frac{x^2+10x}{x^3+x^2-5x+3}dx = \int (x+3)dx + \int \left[\frac{11/4}{(x-1)^2} + \frac{37/16}{x-1} - \frac{21/16}{x+3}\right]dx$$

$$= \int (x+3)dx + (11/4)\int \frac{dx}{(x-1)^2} + (37/16)\int \frac{dx}{x-1} - (21/16)\int \frac{dx}{x+3}$$

$$= \frac{1}{2}x^2 + 3x + \frac{11}{4}\frac{1}{1-x} + \frac{37}{16}\ln(x-1) - \frac{21}{16}\ln(x+3) + C$$